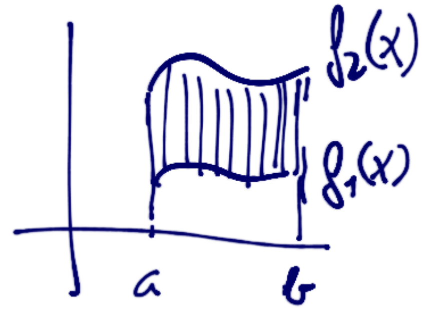
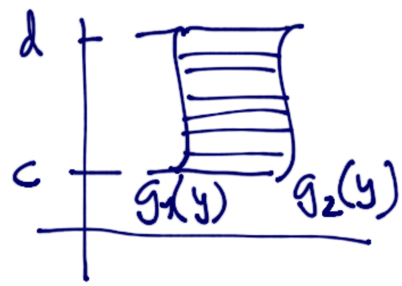
Integration in non-rectangular regions

In 2D

$\boxed{\text{Type I}}$



$\boxed{\text{Type II}}$



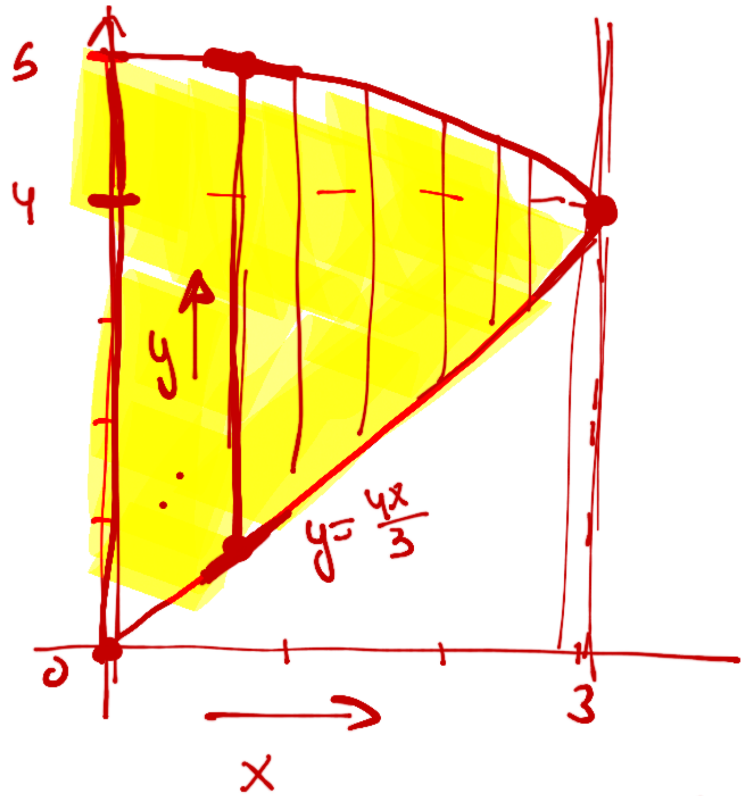
$\boxed{\text{Type 3}}$ Both of them.

Problem 5 – 3.1

i) $$\int_0^3 \int_{4x/3}^{\sqrt{25-x^2}} f(x,y)\, dy\, dx$$

Integration region: (Type I)

$$D = \{ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq 3, \ \frac{4x}{3} \leq y \leq \sqrt{25-x^2} \}$$



$y=\frac{4x}{3}$

$y=\sqrt{25-x^2}$

If $x=3$ $y=4$

If $x=0$, $y=5$

Change the order of integration $\left[\frac{4x}{3}\le y\le\sqrt{25-x^2}\right]$

$\frac{4x}{3}\le y \Rightarrow \boxed{x\le\frac{3y}{4}}$

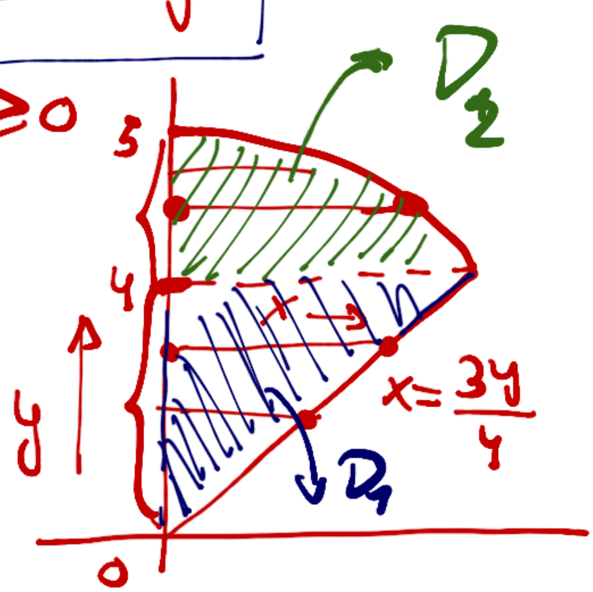
$0\le y\le\sqrt{25-x^2} \Rightarrow \boxed{x\le\sqrt{25-y^2}}$

$x\ge 0$

$0\le x\le\sqrt{25-y^2}$

If $0\le y\le 4$, $0\le x\le\frac{3y}{4}$

If $4\le y\le 5$, $0\le x\le\sqrt{25-y^2}$



New region of integration.

$$D = D_1 \cup D_2 = \underbrace{\left\{(x,y) \in \mathbb{R}^2,\ 0 \le y \le 4,\ 0 \le x \le \frac{3y}{4}\right\}}_{D_1}$$

$$\cup \underbrace{\left\{(x,y) \in \mathbb{R}^2,\ 4 \le y \le 5,\ 0 \le x \le \sqrt{25-y^2}\right\}}_{D_2}$$

$$\int_D f(x,y)\,dy\,dx = \int_{D_1} f(x,y)\,dx\,dy + \int_{D_2} f(x,y)\,dx\,dy.$$

Triple Integrals

Type 1

$$a \le x \le b,\quad \varphi_1(x) \le y \le \varphi_2(x)$$

and $\quad \phi_1(x,y) \le z \le \phi_2(x,y)$

or

$$c \le y \le d,\quad \psi_1(y) \le x \le \psi_2(y)$$

and $\quad \phi_1(x,y) \le z \le \phi_2(x,y)$

$\boxed{\text{Type 2}}$ any other combination.

Change of variables in multiple integration

In linear Algebra you have several transformations

Rotations, $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$(x,y) \to T(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix} = A\bar{X}$$

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

Reflections, projections, ...

We use those transformations to simplify the integrals.

Remember what happens in 1D

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t)) \cdot g'(t)\,dt$$

$$\boxed{x = g(t)}$$

$$\boxed{dx = g'(t)\,dt}$$

If $x = a = g(t)$

$$\Downarrow$$

$$t_0 = g^{-1}(a)$$

$$t_1 = g^{-1}(b)$$

The composition in several variables is similar

$$f(g(t)) \sim f(x,y) \text{ with } \begin{aligned} x &= g_1(s,t) \\ y &= g_2(s,t) \end{aligned}$$

$$f(g_1(s,t), g_2(s,t))$$

However the terms involving the derivative are a bit different.

Definition

Let D^* be a subset in $\mathbb{R}^N$ and

$$T : D^* \subset \mathbb{R}^N \longrightarrow \mathbb{R}^N$$ a transformation of class C^1

We define the Jacobian of the transformation T as

$$JT = \det(DT) = \det \begin{pmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \cdots & \frac{\partial T_1}{\partial x_N} \\ - & - & - & - \\ \frac{\partial T_N}{\partial x_1} & \frac{\partial T_N}{\partial x_2} & \cdots & \frac{\partial T_N}{\partial x_N} \end{pmatrix}$$

In particular in $\mathbb{R}^2$

$T: D^* \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T \in C^1$

$$\left.\begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array}\right\} : T \qquad u,v \text{ new variables.}$$

Then $JT = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Important transformations

In $\mathbb{R}^2$ – Polar coordinates.

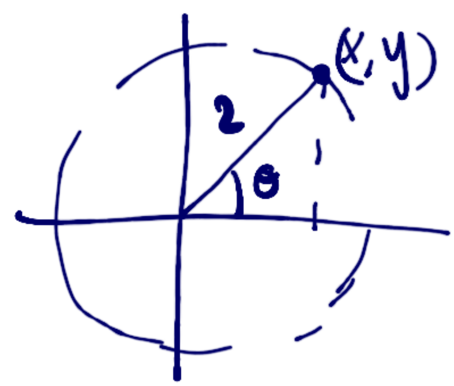
Assume a transformation

$$T : D^* \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(r,\theta) \longrightarrow T(r,\theta) = (x,y)$$

$$= (r\cos\theta, r\sin\theta)$$

$$\boxed{\left.\begin{aligned} x &= \overbrace{r\cos\theta}^{x(r,\theta)} \\ y &= \underbrace{r\sin\theta}_{y(r,\theta)} \end{aligned}\right\} : T}$$



Inverse transformation T^{-1}

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \operatorname{arctg}\frac{y}{x} \end{aligned}\right\}$$

$$JT = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r > 0$$

if $r \neq 0$

and

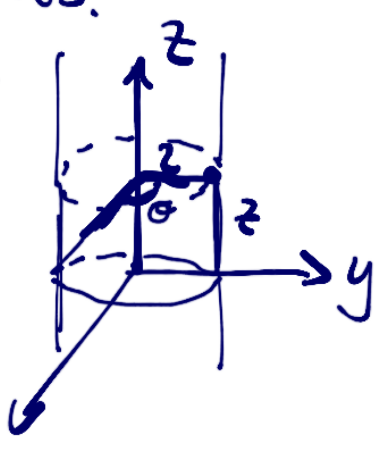
$$D^* = \{(r,\theta) \in \mathbb{R}^2, \ r \geq 0, \ 0 \leq \theta \leq 2\pi\}$$

Since $JT \neq 0$ we can guarantee the existence of the inverse transformation

In $\mathbb{R}^3$ — Cylindrical coordinates.

$(x,y,z) \in \mathbb{R}^3$

$$\boxed{\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} : T}$$



$$D^* = \{(r,\theta,z) \in \mathbb{R}^3 ,\ r \geq 0 ,\ z \in \mathbb{R} ,\ 0 \leq \theta \leq 2\pi\}$$

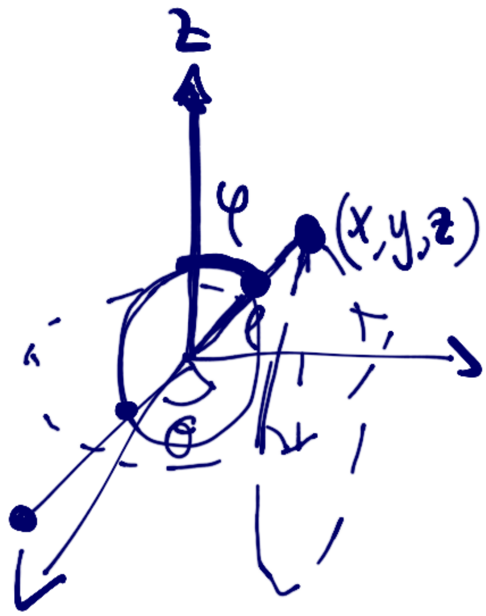
$$JT = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix}$$

$$= \det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r > 0$$

$$\boxed{JT = r}$$

Spherical coordinates.

$(x, y, z) \in \mathbb{R}^3$



$$\left.\begin{aligned} x &= \rho \cos\theta \, \sin\varphi \\ y &= \rho \sin\theta \, \sin\varphi \\ z &= \rho \cos\varphi \end{aligned}\right\} : T$$

$$JT = -\rho^2 \sin\varphi \neq 0$$

we cannot duplicate points.

$$D^* = \{ (\rho, \theta, \varphi) \; ; \; \rho \geq 0 \, , \; 0 \leq \theta \leq 2\pi \, , \; 0 \leq \varphi \leq \pi \}$$

Theorem of change of variables

Let D^* and D be two regions in $\mathbb{R}^N$

and $T : D^* \longrightarrow D$ a transformation.

$\begin{cases} \text{bijective. and of class } C^1 \\ \text{so that } JT \neq 0 \end{cases}$

Then for any integrable function f

$$\int_D f(x_1, x_2, \dots x_N)\, dx_1 \dots dx_N = \int_{D^*} \overbrace{f\big(T(u_1 \dots u_N)\big)}^{(f \circ T)(u_1 \dots u_N)} \underbrace{|JT|}_{\substack{\text{absolute} \\ \text{value of} \\ \text{the } JT}} du_1 \dots du_N$$

Remark

T^{-1} denotes the inverse transformation of T

T is onto and one-to-one, and of class C^1

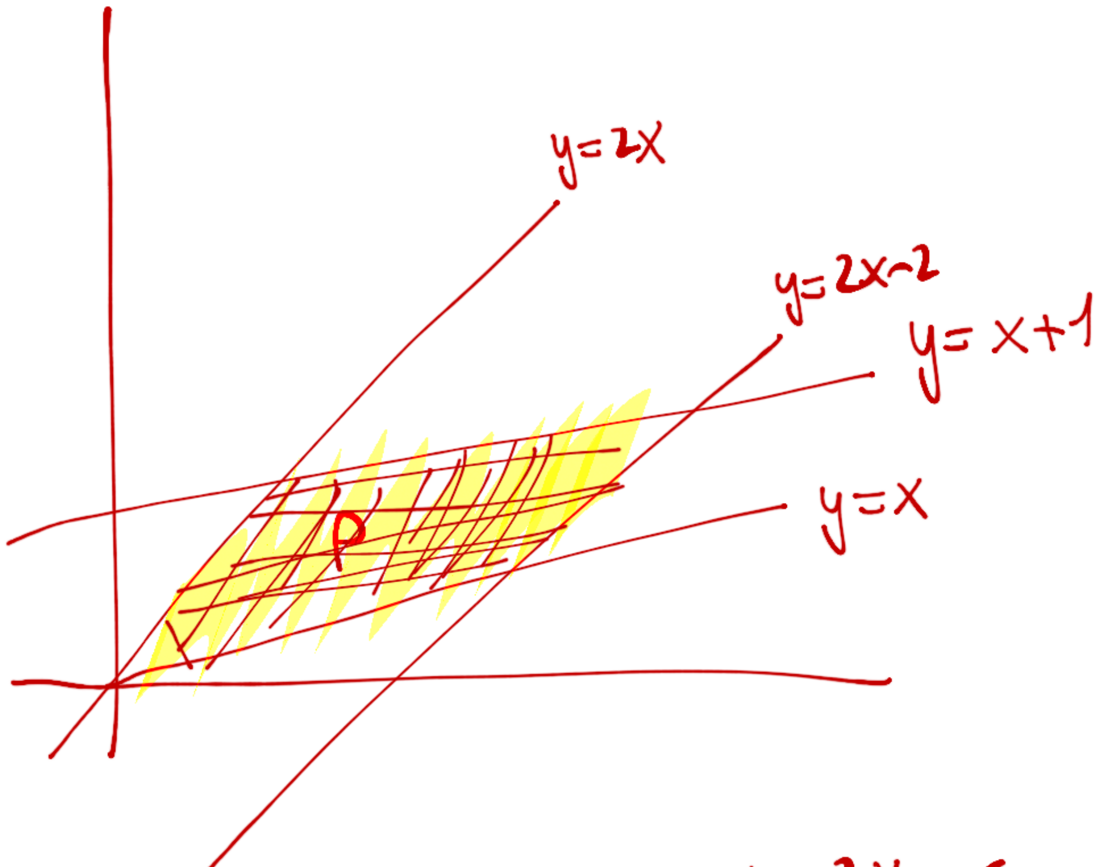
$$JT \neq 0$$

Then $$(JT)\cdot(JT^{-1}) = 1$$

Example: Consider the parallelogram bounded by

$$P : \begin{cases} y = 2x & y = 2x - 2 \\ y = x & y = x + 1 \end{cases}$$

Evaluate : $$\iint_P xy \, dx \, dy$$



$$\begin{cases} y=2x \Rightarrow \underline{y-2x}=0 \\ y=2x-2 \Rightarrow \underline{y-2x}=-2 \end{cases}$$

$$\begin{cases} y=x \Rightarrow \underline{y-x}=0 \\ y=x+1 \Rightarrow \underline{y-x}=1 \end{cases}$$

The natural change of variable

$$T^{-1}: \begin{cases} u=y-2x \text{ such that } -2 \le u \le 0 \\ v=y-x \quad \text{"} \quad \text{"} \quad 0 \le v \le 1 \end{cases}$$

$$u = y - 2x \qquad \left\{ \begin{array}{l} y = v + x \\ u = v + x - 2x \end{array} \right.$$
$$v = y - x$$

$$u - v = -x$$

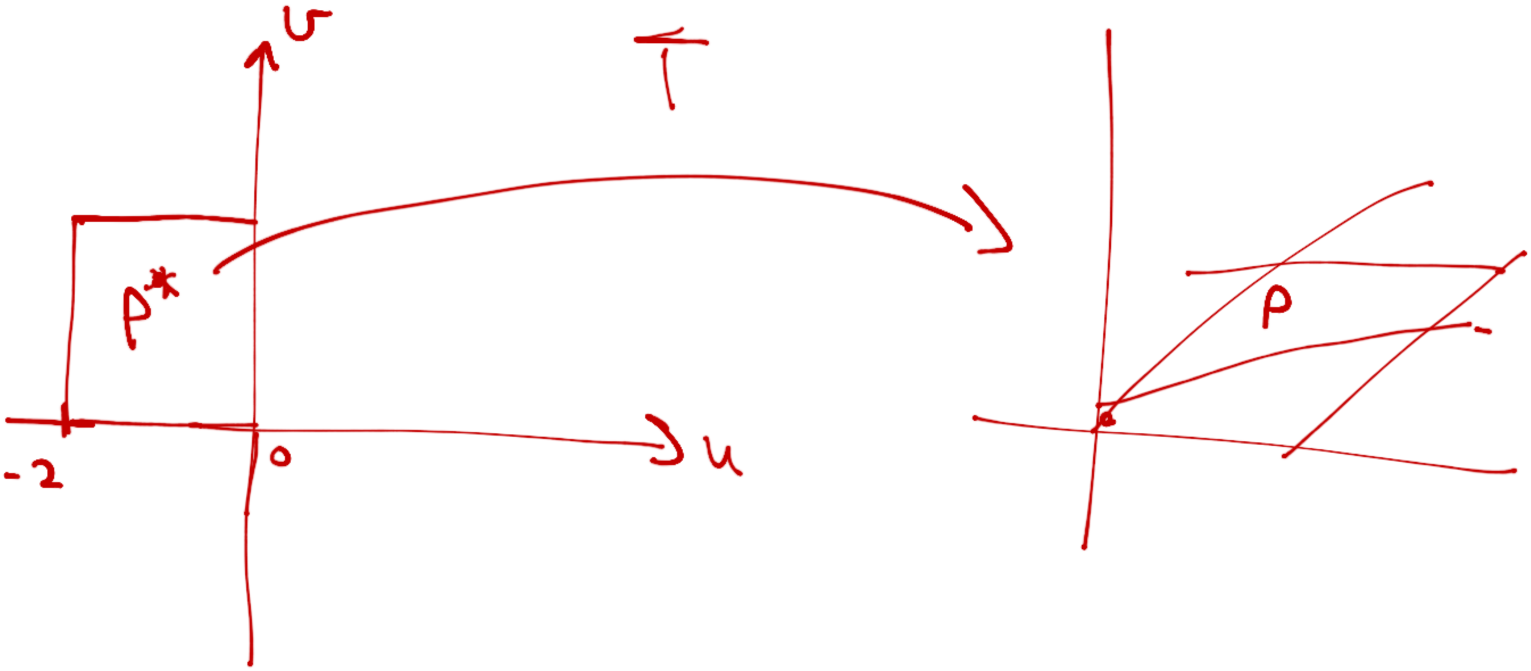
$$T : \left\{ \begin{array}{l} \boxed{x = v - u} \\ \boxed{y = 2v - u} \end{array} \right.$$

$$JT = \det \begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix} = -2 + 1 = -1$$

So that

$$\int_P xy \, dx\, dy = \int_{P^*} (v-u)(2v-u) \cdot \overbrace{|JT|}^{=1} \, du\, dv$$

$$P^* = \{ (u,v) \in \mathbb{R}^2 , \ -2 \le u \le 0 , \ 0 \le v \le 1 \}$$



$$\iint_P xy\,dx\,dy = \int_0^1 \int_{-2}^0 \left(2v^2 + u^2 - 3uv\right) du\,dv$$

$$= \int_0^1 \left(2v^2 u + \frac{u^3}{3} - 3\frac{u^2}{2} v\right)\Big]_{\boxed{-2}}^{0} dv$$

$$= \int_0^1 \left(+4v^2 + \frac{8}{3} + 6v\right) dv = +\frac{4}{3}v^3 + \frac{8}{3}v + 3v^2\Big]_0^1$$

$$= +\frac{4}{3} + \frac{8}{3} + 3$$

Applications of the integral

- Area of a region D

$$A = \iint_D dx\,dy$$

- Volume of Q

$$V = \iiint_Q dx\,dy\,dz$$

in general we call the measure of a region D

$$|D| = \int_D dx_1 \dots dx_N$$

- Mean value for a bounded region D and f integrable.

Mean value or average $= \dfrac{\int_D f}{\text{area of } D}$

The mean value is an extension of what happens for the discrete case.

Assume a number values.

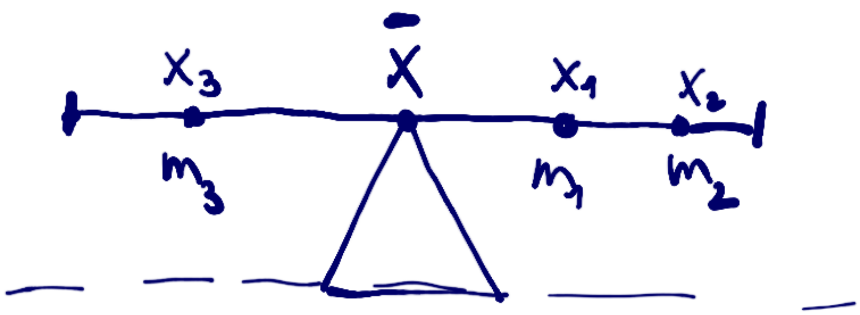
$$x_1 \dots x_n$$

then the mean value will be

$$\frac{x_1+\dots+x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

If we add a mass at each point x_i

Balance of a rod.



$\bar{x} \equiv$ mean value.

$\sum_{i=1}^{n} m_i x_i \equiv$ the addition of those points times the mass

$\sum m_i \equiv$ total mass.

Newton's principle of balance

$$\sum_{i=1}^{n} m_i (x_i - \bar{x}) = 0$$

$$\Downarrow$$

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$ centre of mass

In $\mathbb{R}^3$ and with a continuous density. $\rho(x,y,z)$

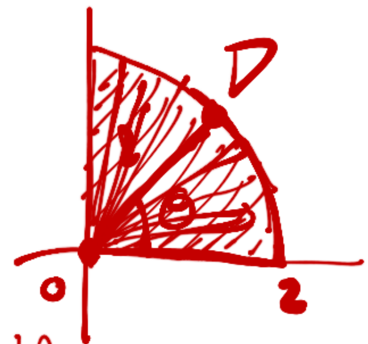
centre of mass.

$$\begin{cases} \bar{x} = \frac{1}{M} \iiint_D x \, \rho(x,y,z) \, dx\,dy\,dz \\ \bar{y} = \frac{1}{M} \iiint_D y \, \rho(x,y,z) \, dx\,dy\,dz \\ \bar{z} = \frac{1}{M} \iiint_D z \, \rho(x,y,z) \, dx\,dy\,dz \end{cases}$$

Total mass.

$$M = \iiint_D \rho(x,y,z)$$

Problem 6 - 3.3



Mass for $D: \{ x^2+y^2 \le 4 \}$, $x,y \ge 0$

density proportional to distance to the origin

$$\rho(x,y) = k\,r = k\sqrt{x^2+y^2} \quad k \equiv \text{constant.}$$

$$M = \iint_D \rho(x,y)\,dx\,dy = \iint_D k\sqrt{x^2+y^2}\,dx\,dy$$

Use polar coordinates change of variable

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} : T \qquad JT = r$$

New region of integration $D^* = \{ (r,\theta) \in \mathbb{R}^2,\ 0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2} \}$

$$M = \iint_D \rho(x,y)\, dx\, dy = \int_0^2 \int_0^{\pi/2} \underbrace{K r}_{} \cdot \underbrace{r}_{|JT|}\, d\theta\, dr$$

$$\sqrt{x^2+y^2} = r \qquad K\sqrt{x^2+y^2}$$

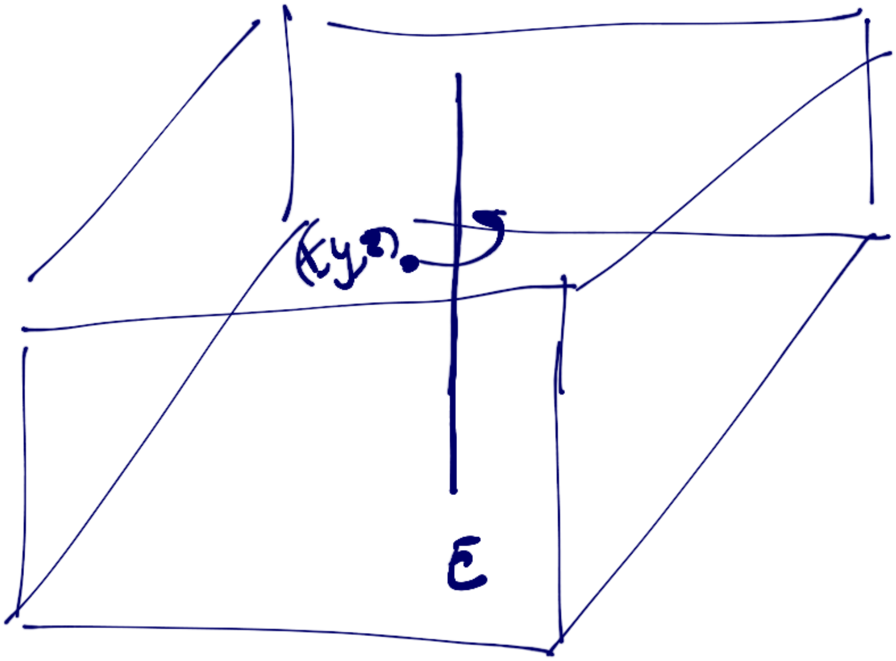
$$= K \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^2 r^2\, dr \right) = \frac{K\pi}{2} \left. \frac{r^3}{3} \right]_0^2 =$$

$$= \frac{K\pi 8}{6} = \frac{4K\pi}{3}$$

Moment of inertia

It measures how a body reacts when turning around an axis. (in 3D)

$$I_E = \iiint_D \left[dist\left((x,y,z), E\right) \right]^2 \underbrace{\rho(x,y,z)}_{\text{density.}}\, dx\, dy\, dz$$



$\mathrm{dist}((x,y,z),E) \equiv$ distance from a point (x,y,z) in D to the axis E

$$\left.\begin{array}{ll} \text{If } E = X & \mathrm{dist}((x,y,z),X) = y^2+z^2 \\ \text{If } E = Y & \mathrm{dist}((x,y,z)\,Y) = x^2+z^2 \\ \text{If } E = Z & \mathrm{dist}((x,y,z)\,Z) = x^2+y^2 \end{array}\right\}$$